MATHEMATICS

ON SOME SPECIAL TYPES OF EXPOSED POINTS OF CLOSED AND BOUNDED SETS IN BANACH SPACES. I

BY

MICHAEL EDELSTEIN

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1. Let S be a closed and bounded set in a metric space (X, d). A point $s \in S$ is said to be boundedly exposed, or to belong to $b(S)$, if and only if an open ball $B = B(c, r) = \{x | d(x, c) < r\}$ exists in $X - \{s\}$ such that $S - \{s\} \subset B$ and $s \in \bar{B} - B$. A point $s \in S$ is said to be σ-boundedly exposed, or to belong to $\sigma b(S)$ if a sequence of open balls $B_n = B(c_n, r_n)$ exists in $X - \{s\}$ with the following properties:

(1.1) $\qquad B_n \subset B_{n+1}, \qquad (n = 1, 2, 3, \ldots);$

(1.2) $\qquad$ the diameter of $S - B_n$ tends to zero as n increases;

(1.3) $\qquad s \in bdry(\bigcup_1^\infty B_n).$

Obviously a boundedly exposed point is σ-boundedly exposed. If X is a normed linear space then, since $\bigcup_1^\infty B_n$ is both open and convex, it follows from (1.3) that any σ-boundedly exposed point of S is actually an exposed point of S.

It is the main purpose of this note to show that in a Banach space satisfying suitable convexity and smoothness conditions $\sigma b(S) \neq \phi$, and in the special case of a Hilbert space even $b(S) \neq \phi$. To this end we now define the two properties of normed spaces that will be used in the hypotheses of Theorem 1.

2. A normed linear space X is said to have property (UD) if for any ε, $0 < \varepsilon \leqslant 2$, there is an $\eta(\varepsilon) > 0$ such that if f is any continuous linear functional on X with $\|f\| = 1$ then the diameter of the set

$$E(f, \delta) = \{x | f(x) \geqslant 1 - \delta \text{ and } \|x\| \leqslant 1\}$$

does not exceed ε whenever $\delta \leqslant \eta(\varepsilon)$; (Cf. [2, p. 113]). RUSTON [5] proved that (UD) is equivalent to uniform convexity [1].

A normed linear space X is said to have property (I) if every bounded closed convex set in X can be represented as the intersection of a family of closed balls; (Cf. [4, p. 976]).

PHELPS [4][1]) strengthened a result of MAZUR [3] according to which all reflexive Banach spaces having a strongly differentiable norm have property (I).

Theorem 1. Suppose X is a Banach space having both property (UD) and (I). If S is a nonempty closed and bounded set in X then $\sigma b(S) \neq \phi$.

Proof. Let $\bar{c}_1$ be an arbitrary point of X and suppose $\bar{r}_1$ is the smallest number for which $C_1 = C(\bar{c}_1, \bar{r}_1) = \{x \mid \|x - \bar{c}_1\| \leqslant \bar{r}_1\}$ contains S. If S is a singleton the theorem is trivially true; otherwise $\bar{r}_1 > 0$. Assuming this let $\delta_1 = \eta\left(\frac{d}{2\bar{r}_1}\right)$, where $d = \operatorname{diam}(S)$, and choose $x_1 \in S$ so that $\|x_1 - \bar{c}_1\| \geqslant r_1\left(1 - \frac{\delta_1}{2}\right)$. Let f_1 be a continuous linear functional with $f_1\left(\frac{\bar{x}_1 - \bar{c}_1}{\|\bar{x}_1 - \bar{c}_1\|}\right) = 1 = \|f_1\|$. If $D_1 = \bar{r}_1\, E(f, \delta) + \bar{c}_1$ then, it is readily seen that $D_1 \subset C_1$, diam $(D_1) \leqslant d/2$ and $x_1 \notin \overline{C_1 - D_1}$. Since $\overline{C_1 - D_1}$ is closed, bounded and convex property (I) implies the existence of $r_1 > 0$ and $c_1 \in X$ such that the open ball $B_1 = B(c_1, r_1) \supset \overline{C_1 - D_1}$ and $x_1 \notin \bar{B}_1$. Since $S - B_1 \subset D_1$ we have diam $(S - B_1) \leqslant d/2$.

Suppose that $B_n = B(c_n, r_n)$ and $x_n \in S$ are already defined so that diam $(S - B_n) \leqslant d/2^n$, $x_n \notin \bar{B}_n$ and (if $n \geqslant 2$) $B_{n-1} \subset B_n$. Then we proceed to define B_{n+1} and x_{n+1} as follows. C_{n+1} will be the closed ball $C(\bar{c}_{n+1}, \bar{r}_{n+1})$ centered at $\bar{c}_{n+1} = c_n$ with $\bar{r}_{n+1}$ the smallest radius for which $C_{n+1} \supset S$; $\delta_{n+1} = \min\left(\frac{d}{2^{n+1}\, \bar{r}_{n+1}}, \frac{\bar{r}_{n+1} - r_n}{\bar{r}_{n+1}}\right)$ and x_{n+1} is a point of S with

$$\|x_{n+1} - \bar{c}_{n+1}\| \geqslant \bar{r}_{n+1}\left(1 - \frac{\delta_{n+1}}{2}\right).$$

If f_{n+1} is a continuous linear functional satisfying

$$f_{n+1}\left(\frac{x_{n+1} - \bar{c}_{n+1}}{\|x_{n+1} - \bar{c}_{n+1}\|}\right) = 1 = \|f_{n+1}\| \quad \text{and} \quad D_{n+1} = \bar{r}_{n+1}\, E(f_{n+1}, \delta_{n+1}) + \bar{c}_{n+1}$$

then, as can be easily seen, $D_{n+1} \subset C_{n+1} - B_n$, $x_{n+1} \notin \overline{C_{n+1} - D_{n+1}}$ and diam $(D_{n+1}) \leqslant d/2^{n+1}$. By (I) there is a closed ball containing $\overline{C_{n+1} - D_{n+1}}$ and not containing x_{n+1}. A concentric B_{n+1} therefore exists such that $x_{n+1} \notin \bar{B}_{n+1} \supset B_{n+1} \supset \overline{C_{n+1} - D_{n+1}}$. Hence diam $(S - B_{n+1}) \leqslant d/2^{n+1}$ and $B_n \subset C_{n+1} - D_{n+1} \subset B_{n+1}$.

The sequence $\{B_n\}$ thus defined satisfies the conditions (1.1) and (1.2). The sequence $\{x_n\}$ is clearly a Cauchy sequence in S which is eventually in $X - B_n$ for $n = 1, 2, \ldots$. It follows from (1.2) that $s = \lim x_n \in bdry(\bigcup_1^\infty B_n)$ so that (1.3) holds too and $s \in \sigma b(S)$ as asserted.

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[1]) I am indebted to Professor L. M. Kelly for this reference.

3. In this section we prove that in Hilbert space $b(S) \neq \phi$. To facilitate the proof we bring the following lemma the verification of which is straightforward and omitted.

Lemma 1. Let U be the open unit ball in the Hilbert space H and $x \in \overline{U} - U$. Suppose $f(z) = <z, x>$, $a > 1$ and $0 < \delta < 1$. Then

$$\left\| y + \frac{x}{a-1} \right\| < \frac{1}{a-1} + 1 - \frac{\delta}{a} \tag{3.1}$$

for all y such that $\|y\| \leqslant 1$ and $f(y) \leqslant 1 - \delta$.

A suitable magnification followed by a translation yields the following.

Corollary. Let $B(c, r) = rU + c$ and $x_2 = r(1 - (\delta/2a)) + c$ then we obtain from (3.1)

$$x_a \notin \overline{B(c', r')} \supset B \subset c', r') \supset \overline{B(c, r) - D} \tag{3.2}$$

where

$$c' = c - \frac{r}{a-1} \frac{x_a}{\|x_a\|}, \quad r' = r\left(\frac{1}{a-1} + 1 - \frac{\delta}{a}\right) \text{ and } D = r\, E(f, \delta) + c,$$

Thus as $s \to \infty$ (and r remains fixed) $\|c' - c\|$ tends to zero in the above configuration.

Theorem 2. Let S be a closed and bounded set in a Hilbert space H. Then $b(S) \neq \phi$.

Proof. Let $\bar{c}_1$, $\bar{r}_1$ and δ_1 be as in the proof of Theorem 1. We choose a_1 large enough so as to have $\bar{r}_1/a_1 - 1 < \frac{1}{2}$ and then $x_1 \in S$ so that

$$\|x_1 - \bar{c}_1\| \geqslant \bar{r}_1(1 - (\delta_1/2a_1)).$$

We then define c_1 and r_1 setting

$$c_1 = \bar{c}_1 - \frac{\bar{r}_1}{a_1 - 1} \frac{x_1}{\|x_1\|} \text{ and } r_1 + \bar{r}_1\left(\frac{1}{a_1 - 1} + 1 - \frac{\delta}{a_1}\right).$$

It follows from the preceding corollary that $\|c_1 - \bar{c}_1\| < \frac{1}{2}$ and

$$x_1 \notin \bar{B}_1 \supset \bar{B}_1 \supset \overline{C_1 - D_1}$$

where, again, $B_1 = B(c_1, r_1)$ $c_1 = C(\bar{c}_1, \bar{r}_1)$, $D_1 = \bar{r}_1 E(f, \delta) + c_1$ and

$$\operatorname{diam}(S - B_1) \leqslant d/2.$$

Suppose that B_n and $x_n \in S$ are already defined so that $\operatorname{diam}(S - B_n) \leqslant d/2^n$, $x_n \notin \bar{B}_n$ and, if $n \geqslant 2$, $B_{n-1} \subset B_n$; in addition let also $\|c_n - c_{n-1}\| < 1/2^n$. Then with C_{n+1} and δ_{n+1} defined as in the proof of Theorem 1 a_{n+1}, x_{n+1}, c_{n+1} and r_{n+1} are defined in terms of $\bar{c}_{n+1} = c_n$, $\bar{r}_{n+1}$ and δ_{n+1} in a manner analogous to that used when defining a_1, x_1, c_1 and r_1 in terms of $\bar{c}_1$, $\bar{r}_1$ and δ_1.

In addition to the properties obtained in section 2 for B_n and x_n the sequence $\{c_n\}$ as presently defined is a Cauchy sequence; hence convergent. It is now clear that if $c = \lim_{n\to\infty} c_n$ and $s = \lim_{n\to\infty} x_n$ then $B(c, \|c-s\|)$ and s satisfy the defining conditions for s to be boundedly exposed.

4. *Remarks*

4.1 Without some convexity and smoothness assumptions on the Banach space X Theorem 1 may fail even for compact S. Indeed if X is the space of all ordered pairs $x=(x_1, x_2)$ of real numbers with $\|x\| = \max(|x_1|, |x_2|)$ and S is the closed unit ball then as can be easily seen $\sigma b(S) = \phi$.

4.2 If S is assumed to be compact strict convexity of X will suffice to assure that $b(S) \neq \phi$. Indeed let $c \in X - S$ be arbitrary and let $r = \sup_S \|c-x\|$. Then the compactness of S implies the existence of $s \in S$ with $\|s-c\| = r$. The ball $B(2c-s, 2r)$ can be seen to satisfy the defining conditions for s to be boundedly exposed.

Dalhousie University, Halifax,
Nova Scotia, Canada

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